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More examples of pseudosymmetric braided categories *

Florin Panaite
Institute of Mathematics of the Romanian Academy
PO-Box 1-764, RO-014700 Bucharest, Romania
e-mail: Florin.Panaite@imar.ro

Mihai D. Staic†
Department of Mathematics and Statistics, BGSU
Bowling Green, OH 43403, USA
e-mail: mstaic@bgsu.edu

Abstract

We study some examples of braided categories and quasitriangular Hopf algebras and decide which of them is pseudosymmetric, respectively pseudotriangular. We show also that there exists a universal pseudosymmetric braided category.

Introduction

Braided categories have been introduced by Joyal and Street in [4] as natural generalizations of symmetric categories. Roughly speaking, a braided category is a category that has a tensor product with a nice commutation rule. More precisely, for every two objects U and V we have an isomorphism $c_{U,V} : U \otimes V \to V \otimes U$ that satisfies certain conditions. These conditions are chosen in such a way that for every object V in the category there exists a natural way to construct a representation for the braid group B_n on $V^{\otimes n}$, therefore the name *braided* categories. If we impose the extra condition $c_{V,U} c_{U,V} = id_{U \otimes V}$ for all objects U, V in the category, we recover the definition of symmetric categories. It is well known that symmetric categories can be used to construct representations for the symmetric group Σ_n.

Pseudosymmetric categories are a special class of braided categories and have been introduced in [9]. The motivation was the study of certain categorical structures called twines, strong twines and pure-braided structures (introduced in [1], [8] and [13]). A braiding on a strict monoidal category is called pseudosymmetric if it satisfies a sort of modified braid relation; any symmetric braiding is pseudosymmetric. One of the most intriguing results obtained in [9] was that the category of Yetter-Drinfeld modules over a Hopf algebra H is pseudosymmetric if and only if H is commutative and cocommutative. We proved in [10] that pseudosymmetric categories can be used to construct representations for the group $PS_n = \frac{B_n}{[P_n, P_n]}$, the quotient of the braid group by the commutator subgroup of the pure braid group. There exists also a Hopf algebraic analogue of pseudosymmetric braidings: a quasitriangular structure on a Hopf algebra is called pseudotriangular if it satisfies a sort of modified quantum Yang-Baxter equation.

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†Institute of Mathematics of the Romanian Academy, PO-Box 1-764, RO-014700 Bucharest, Romania.

In this paper we tie some lose ends from [9] and [10]. We study more examples of braided categories and quasitriangular Hopf algebras and decide when they are pseudosymmetric, respectively pseudotriangular. Namely, we prove that the canonical braiding of the category $\mathcal{LR}(H)$ of Yetter-Drinfeld-Long bimodules over a Hopf algebra H (introduced in [11]) is pseudosymmetric if and only if H is commutative and cocommutative. We show that any quasitriangular structure on the 4ν-dimensional Radford's Hopf algebra H_ν (introduced in [12]) is pseudotriangular. We analyze the positive quasitriangular structures $R(\xi, \eta)$ on a Hopf algebra with positive bases $H(G; G_+, G_-)$ (as defined in [6], [7]), where ξ, η are group homomorphisms from G_+ to G_-, and we present a list of necessary and sufficient conditions for $R(\xi, \eta)$ to be pseudotriangular. If $R(\xi, \eta)$ is normal (i.e. if ξ is trivial) these conditions reduce to the single relation $\eta(uv) = \eta(vu)$ for all $u, v \in G_+$.

In the last section we recall the pseudosymmetric braided category $\mathcal{PS}$ introduced in [10] and we show that it is a universal pseudosymmetric category. More precisely, we prove that it satisfies two universality properties similar to the ones satisfied by the universal braid category $\mathcal{B}$ (see [5]).

1 Preliminaries

We work over a base field k. All algebras, linear spaces, etc, will be over k; unadorned $\otimes$ means $\otimes_k$. For a Hopf algebra H with comultiplication Δ we denote $\Delta(h) = h_1 \otimes h_2$, for $h \in H$. For terminology concerning Hopf algebras and monoidal categories we refer to [5].

Definition 1.1 *([9]) Let $\mathcal{C}$ be a strict monoidal category and c a braiding on $\mathcal{C}$. We say that c is* **pseudosymmetric** *if the following condition holds, for all $X, Y, Z \in \mathcal{C}$:*

$$(c_{Y,Z} \otimes id_X)(id_Y \otimes c^{-1}_{Z,X})(c_{X,Y} \otimes id_Z) = (id_Z \otimes c_{X,Y})(c^{-1}_{Z,X} \otimes id_Y)(id_X \otimes c_{Y,Z}).$$

In this case we say that $\mathcal{C}$ is a **pseudosymmetric braided category***.*

Proposition 1.2 *([9]) Let $\mathcal{C}$ be a strict monoidal category and c a braiding on $\mathcal{C}$. Then c is pseudosymmetric if and only if the family $T_{X,Y} := c_{Y,X}c_{X,Y} : X \otimes Y \to X \otimes Y$ satisfies the condition $(T_{X,Y} \otimes id_Z)(id_X \otimes T_{Y,Z}) = (id_X \otimes T_{Y,Z})(T_{X,Y} \otimes id_Z)$ for all $X, Y, Z \in \mathcal{C}$.*

Definition 1.3 *([9]) Let H be a Hopf algebra and $R \in H \otimes H$ a quasitriangular structure. Then R is called* **pseudotriangular** *if $R_{12}R^{-1}_{31}R_{23} = R_{23}R^{-1}_{31}R_{12}$.*

Proposition 1.4 *([9]) Let H be a Hopf algebra and let R be a quasitriangular structure on H. Then R is pseudotriangular if and only if the element $F = R_{21}R \in H \otimes H$ satisfies the relation $F_{12}F_{23} = F_{23}F_{12}$.*

2 Yetter-Drinfeld-Long bimodules

For a braided monoidal category $\mathcal{C}$ with braiding c, let $\mathcal{C}^{in}$ be equal to $\mathcal{C}$ as a monoidal category, with the mirror-reversed braiding $\tilde{c}_{M,N} := c^{-1}_{N,M}$, for all objects $M, N \in \mathcal{C}$. Directly from the definition of a pseudosymmetric braiding, we immediately obtain:

Proposition 2.1 *Let $\mathcal{C}$ be a strict braided monoidal category. Then $\mathcal{C}$ is pseudosymmetric if and only if $\mathcal{C}^{in}$ is pseudosymmetric.*

Let H be a Hopf algebra with bijective antipode S. Consider the category ${}_H\mathcal{YD}^H$ of left-right Yetter-Drinfeld modules over H, whose objects are vector spaces M that are left H-modules (denote the action by $h\otimes m\mapsto h\cdot m$) and right H-comodules (denote the coaction by $m\mapsto m_{(0)}\otimes m_{(1)}\in M\otimes H$) satisfying the compatibility condition

$$(h\cdot m)_{(0)}\otimes(h\cdot m)_{(1)}=h_2\cdot m_{(0)}\otimes h_3m_{(1)}S^{-1}(h_1),\quad\forall\ h\in H,\ m\in M.$$

It is a monoidal category, with tensor product given by

$$h\cdot(m\otimes n)=h_1\cdot m\otimes h_2\cdot n,\quad(m\otimes n)_{(0)}\otimes(m\otimes n)_{(1)}=m_{(0)}\otimes n_{(0)}\otimes n_{(1)}m_{(1)}.$$

Moreover, it has a (canonical) braiding given by

$$c_{M,N}:M\otimes N\rightarrow N\otimes M,\quad c_{M,N}(m\otimes n)=n_{(0)}\otimes n_{(1)}\cdot m,$$
$$c^{-1}_{M,N}:N\otimes M\rightarrow M\otimes N,\quad c^{-1}_{M,N}(n\otimes m)=S(n_{(1)})\cdot m\otimes n_{(0)}.$$

Consider also the category ${}_H^H\mathcal{YD}$ of left-left Yetter-Drinfeld modules over H, whose objects are vector spaces M that are left H-modules (denote the action by $h\otimes m\mapsto h\cdot m$) and left H-comodules (denote the coaction by $m\mapsto m^{(-1)}\otimes m^{(0)}\in H\otimes M$) with compatibility condition

$$(h_1\cdot m)^{(-1)}h_2\otimes(h_1\cdot m)^{(0)}=h_1m^{(-1)}\otimes h_2\cdot m^{(0)},\quad\forall\ h\in H,\ m\in M.$$

It is a monoidal category, with tensor product given by

$$h\cdot(m\otimes n)=h_1\cdot m\otimes h_2\cdot n,\quad(m\otimes n)^{(-1)}\otimes(m\otimes n)^{(0)}=m^{(-1)}n^{(-1)}\otimes m^{(0)}\otimes n^{(0)}.$$

Moreover, it has a (canonical) braiding given by

$$c_{M,N}:M\otimes N\rightarrow N\otimes M,\quad c_{M,N}(m\otimes n)=m^{(-1)}\cdot n\otimes m^{(0)},$$
$$c^{-1}_{M,N}:N\otimes M\rightarrow M\otimes N,\quad c^{-1}_{M,N}(n\otimes m)=m^{(0)}\otimes S^{-1}(m^{(-1)})\cdot n.$$

Proposition 2.2 *([2]) For the categories ${}_H\mathcal{YD}^H$ and ${}_H^H\mathcal{YD}$ with braidings as above, we have an isomorphism of braided monoidal categories $({}_H\mathcal{YD}^H)^{in}\simeq{}_H^H\mathcal{YD}$.*

Proposition 2.3 *([9]) The canonical braiding of ${}_H\mathcal{YD}^H$ is pseudosymmetric if and only if H is commutative and cocommutative.*

As a consequence of Propositions 2.1, 2.2 and 2.3, we obtain:

Proposition 2.4 *The canonical braiding of ${}_H^H\mathcal{YD}$ is pseudosymmetric if and only if H is commutative and cocommutative.*

We recall now the braided monoidal category $\mathcal{LR}(H)$ defined in [11]. The objects of $\mathcal{LR}(H)$ are vector spaces M endowed with H-bimodule and H-bicomodule structures (denoted by $h\otimes m\mapsto h\cdot m$, $m\otimes h\mapsto m\cdot h$, $m\mapsto m^{(-1)}\otimes m^{(0)}$, $m\mapsto m^{<0>}\otimes m^{<1>}$, for all $h\in H$, $m\in M$), such that M is a left-left Yetter-Drinfeld module, a left-right Long module, a right-right Yetter-Drinfeld module and a right-left Long module, i.e. (for all $h\in H$, $m\in M$):

$$(h_1\cdot m)^{(-1)}h_2\otimes(h_1\cdot m)^{(0)}=h_1m^{(-1)}\otimes h_2\cdot m^{(0)},\tag{2.1}$$

$$(h\cdot m)^{<0>}\otimes(h\cdot m)^{<1>}=h\cdot m^{<0>}\otimes m^{<1>}, \tag{2.2}$$
$$(m\cdot h_2)^{<0>}\otimes h_1(m\cdot h_2)^{<1>}=m^{<0>}\cdot h_1\otimes m^{<1>}h_2, \tag{2.3}$$
$$(m\cdot h)^{(-1)}\otimes(m\cdot h)^{(0)}=m^{(-1)}\otimes m^{(0)}\cdot h. \tag{2.4}$$

Morphisms in $\mathcal{LR}(H)$ are H-bilinear H-bicolinear maps. $\mathcal{LR}(H)$ is a strict monoidal category, with unit k endowed with usual H-bimodule and H-bicomodule structures, and tensor product given by: if $M,N\in\mathcal{LR}(H)$ then $M\otimes N\in\mathcal{LR}(H)$ as follows (for all $m\in M$, $n\in N$, $h\in H$):

$$\begin{aligned}
&h\cdot(m\otimes n)=h_1\cdot m\otimes h_2\cdot n, \quad (m\otimes n)\cdot h=m\cdot h_1\otimes n\cdot h_2,\\
&(m\otimes n)^{(-1)}\otimes(m\otimes n)^{(0)}=m^{(-1)}n^{(-1)}\otimes(m^{(0)}\otimes n^{(0)}),\\
&(m\otimes n)^{<0>}\otimes(m\otimes n)^{<1>}=(m^{<0>}\otimes n^{<0>})\otimes m^{<1>}n^{<1>}.
\end{aligned}$$

Moreover, $\mathcal{LR}(H)$ has a (canonical) braiding defined, for $M,N\in\mathcal{LR}(H)$, $m\in M$, $n\in N$, by

$$\begin{aligned}
&c_{M,N}:M\otimes N\to N\otimes M, \quad c_{M,N}(m\otimes n)=m^{(-1)}\cdot n^{<0>}\otimes m^{(0)}\cdot n^{<1>},\\
&c_{M,N}^{-1}:N\otimes M\to M\otimes N, \quad c_{M,N}^{-1}(n\otimes m)=m^{(0)}\cdot S^{-1}(n^{<1>})\otimes S^{-1}(m^{(-1)})\cdot n^{<0>}.
\end{aligned}$$

Proposition 2.5 *The canonical braiding of $\mathcal{LR}(H)$ is pseudosymmetric if and only if H is commutative and cocommutative.*

Proof. Assume that the canonical braiding of $\mathcal{LR}(H)$ is pseudosymmetric. As noted in [11], ${}_H^H\mathcal{YD}$ with its canonical braiding is a braided subcategory of $\mathcal{LR}(H)$, so the canonical braiding of ${}_H^H\mathcal{YD}$ is pseudosymmetric; by Proposition 2.4 it follows that H is commutative and cocommutative. Conversely, assume that H is commutative and cocommutative. Then one can see that the two Yetter-Drinfeld conditions appearing in the definition of $\mathcal{LR}(H)$ become Long conditions, that is (2.1) and (2.3) become respectively

$$(h\cdot m)^{(-1)}\otimes(h\cdot m)^{(0)}=m^{(-1)}\otimes h\cdot m^{(0)}, \tag{2.5}$$
$$(m\cdot h)^{<0>}\otimes(m\cdot h)^{<1>}=m^{<0>}\cdot h\otimes m^{<1>}. \tag{2.6}$$

Let now $X,Y,Z\in\mathcal{LR}(H)$; we compute, for $x\in X$, $y\in Y$, $z\in Z$:

$$\begin{aligned}
&(c_{Y,Z}\otimes id_X)(id_Y\otimes c_{Z,X}^{-1})(c_{X,Y}\otimes id_Z)(x\otimes y\otimes z)\\
&= (c_{Y,Z}\otimes id_X)(id_Y\otimes c_{Z,X}^{-1})(x^{(-1)}\cdot y^{<0>}\otimes x^{(0)}\cdot y^{<1>}\otimes z)\\
&= (c_{Y,Z}\otimes id_X)(x^{(-1)}\cdot y^{<0>}\otimes z^{(0)}\cdot S^{-1}((x^{(0)}\cdot y^{<1>})^{<1>})\\
&\quad \otimes S^{-1}(z^{(-1)})\cdot(x^{(0)}\cdot y^{<1>})^{<0>})\\
&\overset{(2.6)}{=} (c_{Y,Z}\otimes id_X)(x^{(-1)}\cdot y^{<0>}\otimes z^{(0)}\cdot S^{-1}(x^{(0)<1>})\otimes S^{-1}(z^{(-1)})\cdot x^{(0)<0>}\cdot y^{<1>})\\
&= (x^{(-1)}\cdot y^{<0>})^{(-1)}\cdot[z^{(0)}\cdot S^{-1}(x^{(0)<1>})]^{<0>}\\
&\quad \otimes(x^{(-1)}\cdot y^{<0>})^{(0)}\cdot[z^{(0)}\cdot S^{-1}(x^{(0)<1>})]^{<1>}\\
&\quad \otimes S^{-1}(z^{(-1)})\cdot x^{(0)<0>}\cdot y^{<1>}\\
&\overset{(2.5,2.6)}{=} y^{<0>(-1)}\cdot z^{(0)<0>}\cdot S^{-1}(x^{(0)<1>})\otimes x^{(-1)}\cdot y^{<0>(0)}\cdot z^{(0)<1>}\\
&\quad \otimes S^{-1}(z^{(-1)})\cdot x^{(0)<0>}\cdot y^{<1>},
\end{aligned}$$

$$
\begin{aligned}
&(id_Z \otimes c_{X,Y})(c_{Z,X}^{-1} \otimes id_Y)(id_X \otimes c_{Y,Z})(x \otimes y \otimes z) \\
&= (id_Z \otimes c_{X,Y})(c_{Z,X}^{-1} \otimes id_Y)(x \otimes y^{(-1)} \cdot z^{<0>} \otimes y^{(0)} \cdot z^{<1>}) \\
&= (id_Z \otimes c_{X,Y})([y^{(-1)} \cdot z^{<0>}]^{(0)} \cdot S^{-1}(x^{<1>}) \otimes S^{-1}([y^{(-1)} \cdot z^{<0>}]^{(-1)}) \cdot x^{<0>} \\
&\quad \otimes y^{(0)} \cdot z^{<1>}) \\
&\overset{(2.5)}{=} (id_Z \otimes c_{X,Y})(y^{(-1)} \cdot z^{<0>(0)} \cdot S^{-1}(x^{<1>}) \otimes S^{-1}(z^{<0>(-1)}) \cdot x^{<0>} \otimes y^{(0)} \cdot z^{<1>}) \\
&= y^{(-1)} \cdot z^{<0>(0)} \cdot S^{-1}(x^{<1>}) \otimes [S^{-1}(z^{<0>(-1)}) \cdot x^{<0>}]^{(-1)} \cdot [y^{(0)} \cdot z^{<1>}]^{<0>} \\
&\quad \otimes [S^{-1}(z^{<0>(-1)}) \cdot x^{<0>}]^{(0)} \cdot [y^{(0)} \cdot z^{<1>}]^{<1>} \\
&\overset{(2.5,2.6)}{=} y^{(-1)} \cdot z^{<0>(0)} \cdot S^{-1}(x^{<1>}) \otimes x^{<0>(-1)} \cdot y^{(0)<0>} \cdot z^{<1>} \\
&\quad \otimes S^{-1}(z^{<0>(-1)}) \cdot x^{<0>(0)} \cdot y^{(0)<1>},
\end{aligned}
$$

and the two terms are equal because of the bicomodule condition for X, Y and Z. □

3 Radford's Hopf algebras H_ν

Let ν be an odd natural number and assume that the base field k contains a primitive $2\nu^{th}$ root of unity ω and 2ν is invertible in k. We consider a certain family of Hopf algebras, which are exactly the quasitriangular ones from the larger family constructed by Radford in [12]. Namely, using notation as in [3], we denote by H_ν the Hopf algebra over k generated by two elements g and x such that

$$g^{2\nu} = 1, \quad gx + xg = 0, \quad x^2 = 0,$$

with coproduct $\Delta(g) = g \otimes g$ and $\Delta(x) = x \otimes g^\nu + 1 \otimes x$, and antipode $S(g) = g^{-1}$ and $S(x) = g^\nu x$. Note that H_1 is exactly Sweedler's 4-dimensional Hopf algebra, and in general H_ν is 4ν-dimensional, a linear basis in H_ν being the set $\{g^l x^m / 0 \le l < 2\nu,\ 0 \le m \le 1\}$.

The quasitriangular structures of H_ν have been determined in [12]; they are parametrized by pairs (s, β), where $\beta \in k$ and s is an odd number with $1 \le s < 2\nu$. Moreover, if we denote by $R_{s,\beta}$ the quasitriangular structure corresponding to (s, β), then we have

$$R_{s,\beta} = \frac{1}{2\nu}(\sum_{i,l=0}^{2\nu-1} \omega^{-il} g^i \otimes g^{sl}) + \frac{\beta}{2\nu}(\sum_{i,l=0}^{2\nu-1} \omega^{-il} g^i x \otimes g^{sl+\nu} x).$$

It was also proved in [12] that $R_{s,\beta}$ is triangular if and only if $s = \nu$.

Following [12], we introduce an alternative description of $R_{s,\beta}$, more appropriate for our purpose. For every natural number $0 \le l \le 2\nu - 1$, we define

$$e_l = \frac{1}{2\nu} \sum_{i=0}^{2\nu-1} \omega^{-il} g^i,$$

regarded as an element in the group algebra of the cyclic group of order 2ν generated by the element g (which in turn may be regarded as a Hopf subalgebra of H_ν in the obvious way). Then, by [12], the following relations hold:

$$1 = e_0 + e_1 + ... + e_{2\nu-1},$$

$$
\begin{aligned}
e_i e_j &= \delta_{ij} e_i,\\
g^i e_j &= \omega^{ij} e_j,
\end{aligned}
$$

for all $0 \leq i, j \leq 2\nu - 1$. Also, a straightforward computation shows that we have

$$\sum_{i=0}^{2\nu-1} (-1)^i e_i = g^\nu.$$

Note also that, since ω is a primitive $2\nu^{th}$ root of unity, we have

$$\omega^\nu = -1.$$

With this notation, the quasitriangular structure $R_{s,\beta}$ may be expressed (cf. [12]) as

$$R_{s,\beta} = \sum_{l=0}^{2\nu-1} e_l \otimes g^{sl} + \beta(\sum_{l=0}^{2\nu-1} e_l x \otimes g^{sl+\nu} x).$$

We are interested to see for what s, β is $R_{s,\beta}$ pseudotriangular. We note first that for $\beta = 0$, $R_{s,0}$ is actually a quasitriangular structure on the group algebra of the cyclic group of order 2ν, which is a commutative Hopf algebra, so $R_{s,0}$ is pseudotriangular.

Consider now $R_{s,\beta}$ an arbitrary quasitriangular structure on H_ν. We need to compute first $(R_{s,\beta})_{21} R_{s,\beta}$. By using the defining relations $x^2 = 0$ and $gx + xg = 0$, the properties of the elements e_l listed above and the fact that s and ν are odd numbers, a straightforward computation yields:

$$
\begin{aligned}
(R_{s,\beta})_{21} R_{s,\beta} \quad = \quad & \sum_{l,t=0}^{2\nu-1} \omega^{2slt} e_l \otimes e_t + \beta(\sum_{l,t=0}^{2\nu-1} \omega^{2slt+\nu t} e_l x \otimes e_t x)\\
& -\beta(\sum_{l,t=0}^{2\nu-1} (-1)^{l+t} \omega^{2slt+\nu l} x e_l \otimes e_t x).
\end{aligned}
$$

Let us denote this element by T. We need to compare $T_{12}T_{23}$ and $T_{23}T_{12}$, so we first compute them, using repeatedly the defining relations of H_ν and the properties of the elements e_l:

$$
\begin{aligned}
T_{12}T_{23} \quad = \quad & \sum_{l,t,j=0}^{2\nu-1} \omega^{2slt+2stj} e_l \otimes e_t \otimes e_j + \beta(\sum_{l,t,i,j=0}^{2\nu-1} \omega^{2slt+2sij+\nu j} e_l \otimes e_t e_i x \otimes e_j x\\
& - \sum_{l,t,i,j=0}^{2\nu-1} (-1)^{i+j} \omega^{2slt+2sij+\nu i} e_l \otimes e_t x e_i \otimes e_j x + \sum_{l,t,i,j=0}^{2\nu-1} \omega^{2slt+2sij+\nu t} e_l x \otimes e_t x e_i \otimes e_j\\
& - \sum_{l,t,i,j=0}^{2\nu-1} (-1)^{t+l} \omega^{2slt+2sij+\nu l} x e_l \otimes e_t x e_i \otimes e_j)\\
= \quad & \sum_{l,t,j=0}^{2\nu-1} \omega^{2slt+2stj} e_l \otimes e_t \otimes e_j + \beta(\sum_{l,t,j=0}^{2\nu-1} (-1)^j \omega^{2slt+2stj} e_l \otimes e_t x \otimes e_j x\\
& - \sum_{l,t,i,j=0}^{2\nu-1} (-1)^j \omega^{2slt+2sij} e_l \otimes e_t x e_i \otimes e_j x + \sum_{l,t,i,j=0}^{2\nu-1} (-1)^t \omega^{2slt+2sij} e_l x \otimes e_t x e_i \otimes e_j
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{l,t,i,j=0}^{2\nu-1} (-1)^t \omega^{2slt+2sij} x e_l \otimes e_t x e_i \otimes e_j) \\
= &\sum_{l,t,j=0}^{2\nu-1} \omega^{2slt+2stj} e_l \otimes e_t \otimes e_j + \beta(\sum_{l,t,j=0}^{2\nu-1} (-1)^j \omega^{2slt} e_l \otimes e_t x \otimes g^{2st} e_j x \\
&- \sum_{l,t,i,j=0}^{2\nu-1} (-1)^j g^{2st} e_l \otimes e_t x e_i \otimes g^{2si} e_j x + \sum_{l,t,i,j=0}^{2\nu-1} (-1)^t \omega^{2sij} e_l x \otimes g^{2sl} e_t x e_i \otimes e_j \\
&- \sum_{l,t,i,j=0}^{2\nu-1} (-1)^t \omega^{2sij} x e_l \otimes g^{2sl} e_t x e_i \otimes e_j) \\
= &\sum_{l,t,j=0}^{2\nu-1} \omega^{2slt+2stj} e_l \otimes e_t \otimes e_j + \beta(\sum_{l,t=0}^{2\nu-1} \omega^{2slt} e_l \otimes e_t x \otimes g^{2st+\nu} x \\
&- \sum_{t,i=0}^{2\nu-1} g^{2st} \otimes e_t x e_i \otimes g^{2si+\nu} x + \sum_{l,i,j=0}^{2\nu-1} \omega^{2sij} e_l x \otimes g^{2sl+\nu} x e_i \otimes e_j \\
&- \sum_{l,i,j=0}^{2\nu-1} \omega^{2sij} x e_l \otimes g^{2sl+\nu} x e_i \otimes e_j) \\
= &\sum_{l,t,j=0}^{2\nu-1} \omega^{2slt+2stj} e_l \otimes e_t \otimes e_j + \beta(\sum_{l,t=0}^{2\nu-1} g^{2st} e_l \otimes e_t x \otimes g^{2st+\nu} x \\
&- \sum_{t,i=0}^{2\nu-1} g^{2st} \otimes e_t x e_i \otimes g^{2si+\nu} x + \sum_{l,i,j=0}^{2\nu-1} e_l x \otimes g^{2sl+2sj+\nu} x e_i \otimes e_j \\
&- \sum_{l,i,j=0}^{2\nu-1} x e_l \otimes g^{2sl+2sj+\nu} x e_i \otimes e_j) \\
= &\sum_{l,t,j=0}^{2\nu-1} \omega^{2slt+2stj} e_l \otimes e_t \otimes e_j + \beta(\sum_{t=0}^{2\nu-1} g^{2st} \otimes e_t x \otimes g^{2st+\nu} x \\
&- \sum_{t,i=0}^{2\nu-1} g^{2st} \otimes e_t x e_i \otimes g^{2si+\nu} x + \sum_{l,j=0}^{2\nu-1} e_l x \otimes g^{2sl+2sj+\nu} x \otimes e_j \\
&- \sum_{l,j=0}^{2\nu-1} x e_l \otimes g^{2sl+2sj+\nu} x \otimes e_j),
\end{aligned}
$$

$$
\begin{aligned}
T_{23}T_{12} = &\sum_{l,t,j=0}^{2\nu-1} \omega^{2slt+2stj} e_l \otimes e_t \otimes e_j + \beta(\sum_{l,t,j=0}^{2\nu-1} (-1)^t \omega^{2slt+2stj} e_l x \otimes e_t x \otimes e_j \\
&- \sum_{l,t,j=0}^{2\nu-1} (-1)^t \omega^{2slt+2stj} x e_l \otimes e_t x \otimes e_j + \sum_{l,t,i,j=0}^{2\nu-1} (-1)^j \omega^{2slt+2sij} e_l \otimes e_i x e_t \otimes e_j x \\
&- \sum_{l,t,j=0}^{2\nu-1} (-1)^j \omega^{2slt+2stj} e_l \otimes x e_t \otimes e_j x)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l,t,j=0}^{2\nu-1} \omega^{2slt+2stj} e_l \otimes e_t \otimes e_j + \beta(\sum_{l,t,j=0}^{2\nu-1} (-1)^t e_l x \otimes g^{2sl+2sj} e_t x \otimes e_j \\
&\quad - \sum_{l,t,j=0}^{2\nu-1} (-1)^t x e_l \otimes g^{2sl+2sj} e_t x \otimes e_j + \sum_{l,t,i,j=0}^{2\nu-1} (-1)^j \omega^{2sli+2stj} e_l \otimes e_t x e_i \otimes e_j x \\
&\quad - \sum_{l,t,j=0}^{2\nu-1} (-1)^j \omega^{2slt} e_l \otimes x e_t \otimes g^{2st} e_j x) \\
&= \sum_{l,t,j=0}^{2\nu-1} \omega^{2slt+2stj} e_l \otimes e_t \otimes e_j + \beta(\sum_{l,j=0}^{2\nu-1} e_l x \otimes g^{2sl+2sj+\nu} x \otimes e_j \\
&\quad - \sum_{l,j=0}^{2\nu-1} x e_l \otimes g^{2sl+2sj+\nu} x \otimes e_j + \sum_{l,t,i,j=0}^{2\nu-1} (-1)^j g^{2si} e_l \otimes e_t x e_i \otimes g^{2st} e_j x \\
&\quad - \sum_{l,t=0}^{2\nu-1} g^{2st} e_l \otimes x e_t \otimes g^{2st+\nu} x) \\
&= \sum_{l,t,j=0}^{2\nu-1} \omega^{2slt+2stj} e_l \otimes e_t \otimes e_j + \beta(\sum_{l,j=0}^{2\nu-1} e_l x \otimes g^{2sl+2sj+\nu} x \otimes e_j \\
&\quad - \sum_{l,j=0}^{2\nu-1} x e_l \otimes g^{2sl+2sj+\nu} x \otimes e_j + \sum_{t,i=0}^{2\nu-1} g^{2si} \otimes e_t x e_i \otimes g^{2st+\nu} x \\
&\quad - \sum_{t=0}^{2\nu-1} g^{2st} \otimes x e_t \otimes g^{2st+\nu} x).
\end{aligned}
$$

Thus, we can see that we have

$$
\begin{aligned}
T_{12}T_{23} - T_{23}T_{12} &= \beta(\sum_{t=0}^{2\nu-1} g^{2st} \otimes e_t x \otimes g^{2st+\nu} x - \sum_{t,i=0}^{2\nu-1} g^{2st} \otimes e_t x e_i \otimes g^{2si+\nu} x \\
&\quad - \sum_{t,i=0}^{2\nu-1} g^{2si} \otimes e_t x e_i \otimes g^{2st+\nu} x + \sum_{t=0}^{2\nu-1} g^{2st} \otimes x e_t \otimes g^{2st+\nu} x).
\end{aligned}
$$

We need to prove now that we have

$$x e_l = e_{l-\nu} x,$$

for all $0 \leq l \leq 2\nu - 1$, where the subscripts are taken mod 2ν. We use the following facts:

$$
\begin{aligned}
&\omega^\nu = -1, \\
&x g^i = (-1)^i g^i x = \omega^{i\nu} g^i x.
\end{aligned}
$$

We have:

$$x e_l = x \frac{1}{2\nu} \sum_{i=0}^{2\nu-1} \omega^{-il} g^i$$

$$
\begin{aligned}
&= \frac{1}{2\nu}\sum_{i=0}^{2\nu-1}\omega^{-il}xg^i \\
&= \frac{1}{2\nu}\sum_{i=0}^{2\nu-1}\omega^{-il}\omega^{i\nu}g^ix \\
&= \frac{1}{2\nu}\sum_{i=0}^{2\nu-1}\omega^{-il+i\nu}g^ix \\
&= \frac{1}{2\nu}\sum_{i=0}^{2\nu-1}\omega^{-i(l-\nu)}g^ix \\
&= e_{l-\nu}x, \quad q.e.d.
\end{aligned}
$$

Now we compute:

$$
\begin{aligned}
\sum_{t,i=0}^{2\nu-1} g^{2st}\otimes e_txe_i\otimes g^{2si+\nu}x &= \sum_{t,i=0}^{2\nu-1} g^{2st}\otimes e_te_{i-\nu}x\otimes g^{2si+\nu}x \\
&= \sum_{t,i=0}^{2\nu-1} g^{2st}\otimes \delta_{t,i-\nu}e_tx\otimes g^{2si+\nu}x \\
&= \sum_{t=0}^{2\nu-1} g^{2st}\otimes e_tx\otimes g^{2s(t+\nu)+\nu}x \\
&= \sum_{t=0}^{2\nu-1} g^{2st}\otimes e_tx\otimes g^{2st+\nu}g^{2s\nu}x \\
&= \sum_{t=0}^{2\nu-1} g^{2st}\otimes e_tx\otimes g^{2st+\nu}x,
\end{aligned}
$$

so we have $\sum_{t=0}^{2\nu-1} g^{2st}\otimes e_tx\otimes g^{2st+\nu}x - \sum_{t,i=0}^{2\nu-1} g^{2st}\otimes e_txe_i\otimes g^{2si+\nu}x = 0$. Similarly, we have:

$$
\begin{aligned}
\sum_{t,i=0}^{2\nu-1} g^{2si}\otimes e_txe_i\otimes g^{2st+\nu}x &= \sum_{t,i=0}^{2\nu-1} g^{2si}\otimes xe_{t+\nu}e_i\otimes g^{2st+\nu}x \\
&= \sum_{t,i=0}^{2\nu-1} g^{2si}\otimes x\delta_{t+\nu,i}e_i\otimes g^{2st+\nu}x \\
&= \sum_{i=0}^{2\nu-1} g^{2si}\otimes xe_i\otimes g^{2s(i-\nu)+\nu}x \\
&= \sum_{i=0}^{2\nu-1} g^{2si}\otimes xe_i\otimes g^{2si+\nu}x,
\end{aligned}
$$

so we have $\sum_{t=0}^{2\nu-1} g^{2st}\otimes xe_t\otimes g^{2st+\nu}x - \sum_{t,i=0}^{2\nu-1} g^{2si}\otimes e_txe_i\otimes g^{2st+\nu}x = 0$. Consequently, we have $T_{12}T_{23} - T_{23}T_{12} = 0$, and so we obtained:

Theorem 3.1 *Any quasitriangular structure $R_{s,\beta}$ on Radford's Hopf algebra H_ν is pseudotriangular.*

4 Hopf algebras with positive bases

In this section the base field is assumed to be $\mathbb{C}$, the field of complex numbers.

We recall from [6] that a basis of a Hopf algebra over $\mathbb{C}$ is called *positive* if all the structure constants (for the unit, counit, multiplication, comultiplication and antipode) with respect to this basis are nonnegative real numbers. Also, a quasitriangular structure R on a Hopf algebra having a positive basis B is called *positive* in [7] if the coefficients of R in the basis $B \otimes B$ are nonnegative real numbers. The finite dimensional Hopf algebras having a positive basis and the positive quasitriangular structures on them have been classified in [6], [7] as follows.

Let G be a group (we denote by e its unit). A *unique factorization* $G = G_+G_-$ of G consists of two subgroups G_+ and G_- of G such that any $g \in G$ can be written uniquely as $g = g_+g_-$, with $g_+ \in G_+$ and $g_- \in G_-$. By considering the inverse map, we can also write uniquely $g = \overline{g}_-\overline{g}_+$, with $\overline{g}_- \in G_-$ and $\overline{g}_+ \in G_+$.

Let $u \in G_+$, $x \in G_-$; then we can write uniquely

$$xu = ({}^xu)(x^u), \ \ \textit{with} \ \ {}^xu \in G_+ \ \ \textit{and} \ \ x^u \in G_-,$$
$$ux = ({}^ux)(u^x), \ \ \textit{with} \ \ {}^ux \in G_- \ \ \textit{and} \ \ u^x \in G_+.$$

So, we have the following actions of G_+ and G_- on each other (from left and right):

$$\begin{aligned} G_- \times G_+ &\to G_+, \quad (x,u) \mapsto {}^xu, \\ G_- \times G_+ &\to G_-, \quad (x,u) \mapsto x^u, \\ G_+ \times G_- &\to G_-, \quad (u,x) \mapsto {}^ux, \\ G_+ \times G_- &\to G_+, \quad (u,x) \mapsto u^x. \end{aligned}$$

The relations between these actions and the decompositions $g = g_+g_- = \overline{g}_-\overline{g}_+$ are:
${}^{\overline{g}_-}\overline{g}_+ = g_+$; $\overline{g}_-^{\overline{g}_+} = g_-$; $g_+^{g_-} = \overline{g}_+$; ${}^{g_+}g_- = \overline{g}_-$; $({}^{g_+}g_-)(g_+^{g_-}) = g_+g_-$; $({}^{g_-}g_+)(g_-^{g_+}) = g_-g_+$.

Given a unique factorization $G = G_+G_-$ of a finite group G, one can construct a finite dimensional Hopf algebra $H(G;G_+,G_-)$, which is the vector space spanned by the set G (we denote by $\{g\}$ an element $g \in G$ when it is regarded as an element in $H(G;G_+,G_-)$) with the following Hopf algebra structure:
multiplication: $\{g\}\{h\} = \delta_{g_+^{g_-},h_+}\{gh_-\}$
unit: $1 = \sum_{g_+ \in G_+}\{g_+\}$
comultiplication: $\Delta(\{g\}) = \sum_{h_+ \in G_+}\{g_+h_+^{-1}({}^{h_+}g_-)\} \otimes \{h_+g_-\}$
counit: $\varepsilon(\{g\}) = \delta_{g_+,e}$
antipode: $S(\{g\}) = \{g^{-1}\}$

The Hopf algebra $H(G;G_+,G_-)$ has G as the obvious positive basis. Conversely, it was proved in [6] that all finite dimensional Hopf algebras with positive bases are of the form $H(G;G_+,G_-)$.

The positive quasitriangular and triangular structures on $H(G;G_+,G_-)$ have been described in [7] as follows:

Theorem 4.1 *([7]) Let $G = G_+G_-$ be a unique factorization of a finite group G. Let $\xi, \eta : G_+ \to G_-$ be two group homomorphisms satisfying the following conditions:*

$$\xi(u)^v = \xi(u^{\eta(v)}), \tag{4.1}$$

$$\begin{aligned}
&{}^{u}\eta(v) = \eta({}^{\xi(u)}v), && (4.2)\\
&uv = ({}^{\xi(u)}v)(u^{\eta(v)}), && (4.3)\\
&\xi({}^{x}u)x^{u} = x\xi(u), && (4.4)\\
&\eta({}^{x}u)x^{u} = x\eta(u), && (4.5)
\end{aligned}$$

for all $u, v \in G_+$ and $x \in G_-$. Then

$$R(\xi,\eta) := \sum_{u,v\in G_+} \{u(\eta(v)^u)^{-1}\} \otimes \{v\xi(u)\}$$

is a positive quasitriangular structure on $H(G; G_+, G_-)$. Conversely, every positive quasitriangular structure on $H(G; G_+, G_-)$ is given by the above construction.

Moreover, each of the conditions (4.1)-(4.5) is equivalent to the corresponding property below:

$$\begin{aligned}
&{}^{v}\xi(u) = \xi({}^{\eta(v)}u), && (4.6)\\
&\eta(v)^{u} = \eta(v^{\xi(u)}), && (4.7)\\
&uv = ({}^{\eta(u)}v)(u^{\xi(v)}), && (4.8)\\
&{}^{u}x\xi(u^{x}) = \xi(u)x, && (4.9)\\
&{}^{u}x\eta(u^{x}) = \eta(u)x. && (4.10)
\end{aligned}$$

Moreover, $R(\xi,\eta)$ is triangular if and only if $\xi = \eta$.

Our aim now is to characterize those $R(\xi,\eta)$ that are pseudotriangular. So, let $R = R(\xi,\eta)$ be a positive quasitriangular structure on $H(G; G_+, G_-)$. We have (see [7]):

$$R_{21}R = \sum_{u,v\in G_+} \{v\xi(u)(\eta(\overline{v})^{\overline{u}})^{-1}\} \otimes \{u(\eta(v)^u)^{-1}\xi(\overline{u})\},$$

where we denoted $\overline{u} = v^{\xi(u)}$ and $\overline{v} = {}^{\eta(v)}u$.

We denote $T = R_{21}R$ and we compute (by using the formula for the multiplication of $H(G; G_+, G_-)$):

$$\begin{aligned}
T_{12}T_{23} &= \Big(\sum_{u,v\in G_+} \{v\xi(u)(\eta(\overline{v})^{\overline{u}})^{-1}\} \otimes \{u(\eta(v)^u)^{-1}\xi(\overline{u})\} \otimes 1\Big)\\
&\quad \Big(\sum_{s,t\in G_+} 1 \otimes \{t\xi(s)(\eta(\overline{t})^{\overline{s}})^{-1}\} \otimes \{s(\eta(t)^s)^{-1}\xi(\overline{s})\}\Big)\\
&= \sum_{u,v,s,t\in G_+} \{v\xi(u)(\eta(\overline{v})^{\overline{u}})^{-1}\} \otimes \{u(\eta(v)^u)^{-1}\xi(\overline{u})\}\{t\xi(s)(\eta(\overline{t})^{\overline{s}})^{-1}\}\\
&\quad \otimes\{s(\eta(t)^s)^{-1}\xi(\overline{s})\}\Big)\\
&= \sum_{u,v,s\in G_+} \{v\xi(u)(\eta({}^{\eta(v)}u)^{(v^{\xi(u)})})^{-1}\} \otimes \{u(\eta(v)^u)^{-1}\xi(v^{\xi(u)})\xi(s)(\eta({}^{\eta(t)}s)^{(t^{\xi(s)})})^{-1}\}\\
&\quad \otimes\{s(\eta(t)^s)^{-1}\xi(t^{\xi(s)})\},
\end{aligned}$$

where $t = u^{(\eta(v)^u)^{-1}\xi(v^{\xi(u)})}$, and

$$T_{23}T_{12} = \Big(\sum_{a,b\in G_+} 1 \otimes \{b\xi(a)(\eta({}^{\eta(b)}a)^{(b^{\xi(a)})})^{-1}\} \otimes \{a(\eta(b)^a)^{-1}\xi(b^{\xi(a)})\}\Big)$$

$$
\begin{aligned}
&\Big(\sum_{c,d\in G_+}\{d\xi(c)(\eta({}^{\eta(d)}c)^{(d^{\xi(c)})})^{-1}\}\otimes\{c(\eta(d)^c)^{-1}\xi(d^{\xi(c)})\}\otimes 1\Big)\\
=&\sum_{a,b,c,d\in G_+}\{d\xi(c)(\eta({}^{\eta(d)}c)^{(d^{\xi(c)})})^{-1}\}\otimes\{b\xi(a)(\eta({}^{\eta(b)}a)^{(b^{\xi(a)})})^{-1}\}\{c(\eta(d)^c)^{-1}\xi(d^{\xi(c)})\}\\
&\otimes\{a(\eta(b)^a)^{-1}\xi(b^{\xi(a)})\}\\
=&\sum_{a,b,d\in G_+}\{d\xi(c)(\eta({}^{\eta(d)}c)^{(d^{\xi(c)})})^{-1}\}\otimes\{b\xi(a)(\eta({}^{\eta(b)}a)^{(b^{\xi(a)})})^{-1}(\eta(d)^c)^{-1}\xi(d^{\xi(c)})\}\\
&\otimes\{a(\eta(b)^a)^{-1}\xi(b^{\xi(a)})\},
\end{aligned}
$$

where $c = b^{\xi(a)(\eta({}^{\eta(b)}a)^{(b^{\xi(a)})})^{-1}}$. By writing down what means $T_{12}T_{23} = T_{23}T_{12}$, we obtain:

Proposition 4.2 *The positive quasitriangular structure $R(\xi,\eta)$ is pseudotriangular if and only if the following conditions are satisfied:*

$$
\begin{aligned}
&\xi(u)(\eta({}^{\eta(v)}u)^{(v^{\xi(u)})})^{-1} = \xi(c)(\eta({}^{\eta(v)}c)^{(v^{\xi(c)})})^{-1},\\
&(\eta(v)^u)^{-1}\xi(v^{\xi(u)})\xi(s)(\eta({}^{\eta(t)}s)^{(t^{\xi(s)})})^{-1} = \xi(s)(\eta({}^{\eta(u)}s)^{(u^{\xi(s)})})^{-1}(\eta(v)^c)^{-1}\xi(v^{\xi(c)}),\\
&(\eta(t)^s)^{-1}\xi(t^{\xi(s)}) = (\eta(u)^s)^{-1}\xi(u^{\xi(s)}),
\end{aligned}
$$

for all $u,v,s\in G_+$, where $t = u^{(\eta(v)^u)^{-1}\xi(v^{\xi(u)})}$ and $c = u^{\xi(s)(\eta({}^{\eta(u)}s)^{(u^{\xi(s)})})^{-1}}$.

A better description may be obtained for a certain class of positive quasitriangular structures.

Definition 4.3 *([7]) A positive quasitriangular structure $R(\xi,\eta)$ on $H(G;G_+,G_-)$ is called* normal *if $\xi(u)=e$ for all $u\in G_+$.*

Theorem 4.4 *A normal positive quasitriangular structure $R(\xi,\eta)$ on $H(G;G_+,G_-)$ is pseudotriangular if and only if $\eta(uv)=\eta(vu)$ for all $u,v\in G_+$.*

Proof. We note first that, since $\xi(u)=e$ for all $u\in G_+$, some of the relations (4.1)-(4.10) may be simplified, in particular we have ${}^u\eta(v)=\eta(v)$, $uv = v(u^{\eta(v)})$, $\eta(v)^u=\eta(v)$, $uv=({}^{\eta(u)}v)u$, for all $u,v\in G_+$. By using these relations, together with the fact that $\xi(u)=e$ for all $u\in G_+$, the three conditions in the above Proposition may be also simplified, so we obtain that $R(\xi,\eta)$ is pseudotriangular if and only if we have:

$$
\begin{aligned}
&\eta(vuv^{-1}) = \eta(vcv^{-1}),\\
&\eta(v)^{-1}\eta(tst^{-1})^{-1} = \eta(usu^{-1})^{-1}\eta(v)^{-1},\\
&\eta(t)^{-1} = \eta(u)^{-1},
\end{aligned}
$$

for all $u,v,s\in G_+$, where $t=vuv^{-1}$ and $c=usus^{-1}u^{-1}$, and one can easily see that each of these three conditions is equivalent to the condition $\eta(uv)=\eta(vu)$, for all $u,v\in G_+$. $\square$

We recall from [9] that the canonical quasitriangular structure on the Drinfeld double of a finite dimensional Hopf algebra H is pseudotriangular if and only if H is commutative and cocommutative. In particular, if G is a finite group, the canonical quasitriangular structure on the Drinfeld double of the dual $k[G]^*$ of the group algebra $k[G]$ is pseudotriangular if and only if G is abelian. We want to reobtain this result (over $\mathbb{C}$) as an application of Theorem 4.4.

We consider the unique factorization $G = G_+G_-$, where $G_+ = G$ and $G_- = \{e\}$ (so the Hopf algebra $H(G; G_+, G_-)$ is exactly $k[G]^*$). As in [7], we consider the group $\tilde{G} = G \times G$, with the unique factorization $\tilde{G} = \tilde{G}_+\tilde{G}_-$, where $\tilde{G}_+ = G \times \{e\}$ and $\tilde{G}_- = \{(g, g) : g \in G\}$. By [7], the group homomorphisms $\xi, \eta : \tilde{G}_+ \to \tilde{G}_-$ defined by $\xi(g, e) = (e, e)$ and $\eta(g, e) = (g, g)$ induce a positive quasitriangular structure $R(\xi, \eta)$ on $H(\tilde{G}; \tilde{G}_+, \tilde{G}_-)$ and moreover $H(\tilde{G}; \tilde{G}_+, \tilde{G}_-)$ is the Drinfeld double of $H(G; G_+, G_-) = k[G]^*$ and $R(\xi, \eta)$ is its canonical quasitriangular structure. Obviously $R(\xi, \eta)$ is normal, so we may apply Theorem 4.4 and we obtain that $R(\xi, \eta)$ is pseudotriangular if and only if $(gh, gh) = (hg, hg)$ for all $g, h \in G$, i.e. if and only if G is abelian.

5 Universality of the pseudosymmetric category $\mathcal{PS}$

In this section we use terminology, notation and some results from [5] (but we use the term "monoidal" instead of "tensor" when we speak about tensor categories and tensor functors).

Our aim is to show that the pseudosymmetric category $\mathcal{PS}$ introduced in [10] has two universality properties similar to the ones of the braid category $\mathcal{B}$, the universal braided monoidal category (see [5]). First, we recall from [10] the definition of $\mathcal{PS}$. The objects of $\mathcal{PS}$ are natural numbers $n \in \mathbb{N}$. The set of morphisms from m to n is empty if $m \neq n$ and is $PS_n := \frac{B_n}{[P_n, P_n]}$ if $m = n$, where B_n (respectively P_n) is the braid group (respectively pure braid group) on n strands. The monoidal structure of $\mathcal{PS}$ is defined as the one for $\mathcal{B}$, and so is the braiding, namely (we denote as usual by $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ the standard generators of B_n and by π_n the natural morphism from B_n to PS_n):

$$c_{n,m} : n \otimes m \to m \otimes n, \quad c_{0,n} = id_n = c_{n,0},$$
$$c_{n,m} = \pi_{n+m}((\sigma_m\sigma_{m-1} \cdots \sigma_1)(\sigma_{m+1}\sigma_m \cdots \sigma_2) \cdots (\sigma_{m+n-1}\sigma_{m+n-2} \cdots \sigma_n)) \text{ if } m, n > 0.$$

In order to introduce the first universality property for $\mathcal{PS}$, we need the following definition, motivated by results in [10] and by the definition of Yang-Baxter operators from [5]:

Definition 5.1 *If V is an object in a monoidal category $(\mathcal{C}, \otimes, I, a, l, r)$, an automorphism σ of $V \otimes V$ is called a* pseudosymmetric Yang-Baxter operator *on V if the following two dodecagons*

(for σ and σ^{-1}) commute:

$$\begin{tikzcd}
 & (V\otimes V)\otimes V \arrow[dl, "\sigma\otimes id_V"'] \arrow[dr, "a_{V,V,V}"] & \\
(V\otimes V)\otimes V \arrow[d, "a_{V,V,V}"'] & & V\otimes(V\otimes V) \arrow[d, "id_V\otimes\sigma"] \\
V\otimes(V\otimes V) \arrow[d, "id_V\otimes\sigma^{\pm 1}"'] & & V\otimes(V\otimes V) \arrow[d, "a^{-1}_{V,V,V}"] \\
V\otimes(V\otimes V) \arrow[d, "a^{-1}_{V,V,V}"'] & & (V\otimes V)\otimes V \arrow[d, "\sigma^{\pm 1}\otimes id_V"] \\
(V\otimes V)\otimes V \arrow[d, "\sigma\otimes id_V"'] & & (V\otimes V)\otimes V \arrow[d, "a_{V,V,V}"] \\
(V\otimes V)\otimes V \arrow[dr, "a_{V,V,V}"'] & & V\otimes(V\otimes V) \arrow[dl, "id_V\otimes\sigma"] \\
 & V\otimes(V\otimes V) &
\end{tikzcd}$$

Note that a pseudosymmetric Yang-Baxter operator is a special type of Yang-Baxter operator as defined in [5], p. 323. Moreover, just like Yang-Baxter operators, they can be transferred by using functors between monoidal categories:

Lemma 5.2 *Let $(F, \varphi_0, \varphi_2) : \mathcal{C} \to \mathcal{D}$ be a monoidal functor between two monoidal categories. If $\sigma \in Aut(V \otimes V)$ is a pseudosymmetric Yang-Baxter operator on the object $V \in \mathcal{C}$, then*

$$\sigma' = \varphi_2(V, V)^{-1} \circ F(\sigma) \circ \varphi_2(V, V)$$

is a pseudosymmetric Yang-Baxter operator on $F(V)$.

Proof. The proof follows exactly as in [5], Lemma XIII.3.2, by using also the identity

$$(\sigma')^{-1} = \varphi_2(V, V)^{-1} \circ F(\sigma^{-1}) \circ \varphi_2(V, V)$$

in order to prove the pseudosymmetry of σ'. $\square$

We define the category $PSYB(\mathcal{C})$ of pseudosymmetric Yang-Baxter operators to be a full subcategory of $YB(\mathcal{C})$, the category of Yang-Baxter operators defined in [5]. An object in $PSYB(\mathcal{C})$ is a pair (V, σ) where V is a object in $\mathcal{C}$ and σ is a pseudosymmetric Yang-Baxter operator.

Recall the following construction from [5]. Suppose that $(F, \varphi_0, \varphi_2) : \mathcal{B} \to \mathcal{C}$ is a monoidal functor from the universal braid category $\mathcal{B}$ to a given monoidal category $\mathcal{C}$. Since $c_{1,1} = \sigma_1$ is a Yang-Baxter operator on the object $1 \in \mathcal{B}$, it follows that $\sigma = \varphi_2^{-1}(1, 1)F(c_{1,1})\varphi_2(1, 1)$ is a Yang-Baxter operator on $F(1) \in \mathcal{C}$. In this way we get a functor $\Theta : Tens(\mathcal{B}, \mathcal{C}) \to YB(\mathcal{C})$, where $Tens(\mathcal{B}, \mathcal{C})$ is the category of monoidal functors from $\mathcal{B}$ to $\mathcal{C}$. It was proved in [5] that:

Theorem 5.3 *([5]) For any monoidal category $\mathcal{C}$, the functor $\Theta : Tens(\mathcal{B}, \mathcal{C}) \to YB(\mathcal{C})$ is an equivalence of categories.*

One can note that we have a natural monoidal functor $\pi : \mathcal{B} \to \mathcal{PS}$ induced by the group epimorphism $\pi_n : B_n \to PS_n$. This allows us to identify the category $Tens(\mathcal{PS}, \mathcal{C})$ with a subcategory of $Tens(\mathcal{B}, \mathcal{C})$. More precisely, we identify it with the full subcategory of all monoidal functors $F : \mathcal{B} \to \mathcal{C}$ with the property that there exists a monoidal functor $G : \mathcal{PS} \to \mathcal{C}$ such that $F = G \circ \pi$.

We can state now the first universality property of $\mathcal{PS}$:

Theorem 5.4 *For any monoidal category $\mathcal{C}$, the functor $\widetilde{\Theta} : Tens(\mathcal{PS}, \mathcal{C}) \to PSYB(\mathcal{C})$, $\widetilde{\Theta}(G) = \Theta(G \circ \pi)$ is an equivalence of categories.*

Proof. First we note that $\pi(c_{1,1})$ is a pseudosymmetric Yang-Baxter operator in $\mathcal{PS}$ and so by Lemma 5.2 we have $\varphi_2^{-1}(1,1)G(\pi(c_{1,1}))\varphi_2(1,1) \in PSYB(\mathcal{C})$. This means that $\widetilde{\Theta}$ is well defined. Since Θ is fully faithful and $\widetilde{\Theta}$ is its restriction to a full subcategory, it is enough to show that $\widetilde{\Theta}$ is essentially surjective. This follows from the next lemma. $\square$

Lemma 5.5 *Let $\mathcal{C}$ be a strict monoidal category and (V, σ) an object in $PSYB(\mathcal{C})$. Then there exists a unique strict monoidal functor $G : \mathcal{PS} \to \mathcal{C}$ such that $G(1) = V$ and $G(\pi(c_{1,1})) = \sigma$.*

Proof. From [5], Lemma XIII.3.5 we know that for all $(V, \sigma) \in YB(\mathcal{C})$ there exists a unique strict monoidal functor $F : \mathcal{B} \to \mathcal{C}$ such that $F(1) = V$ and $F(c_{1,1}) = \sigma$. It is enough to show that when $(V, \sigma) \in PSYB(\mathcal{C})$ the functor F factors through π. But this follows immediately from the fact (see [10]) that

$$PS_n = \frac{B_n}{< \sigma_i \sigma_{i+1}^{-1} \sigma_i = \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1} | 1 \leq i \leq n-2 >}$$

and the definition of a pseudosymmetric Yang-Baxter operator. $\square$

Definition 5.6 *([5]) A monoidal functor $(F, \varphi_0, \varphi_2)$ from a braided monoidal category $\mathcal{C}$ to a braided monoidal category $\mathcal{D}$ is braided if for every pair (U, V) of objects in $\mathcal{C}$ the square*

$$\begin{array}{ccc} F(U) \otimes F(V) & \xrightarrow{\varphi_2} & F(U \otimes V) \\ {\scriptstyle c_{F(U),F(V)}} \downarrow & & \downarrow {\scriptstyle F(c_{U,V})} \\ F(V) \otimes F(U) & \xrightarrow{\varphi_2} & F(V \otimes U) \end{array}$$

commutes. Denote by $Br(\mathcal{C}, \mathcal{D})$ the category whose objects are braided monoidal functors and morphisms are natural monoidal transformations.

Theorem 5.7 *([5]) For a braided monoidal category $\mathcal{C}$, the functor $\Theta' : Br(\mathcal{B}, \mathcal{C}) \to \mathcal{C}$ defined by $\Theta'(F) = F(1)$ is an equivalence of categories.*

In the definition of a pseudosymmetric braided category $\mathcal{C}$ introduced in [9] was assumed that $\mathcal{C}$ was a *strict* monoidal category. The next proposition is the analogue of Theorem 3.7 from [9] for monoidal categories with nontrivial associativity constraints. Note that the proof that we present here is very direct and is inspired by the results in [10].

Proposition 5.8 *Let $(\mathcal{C}, \otimes, I, a, l, r, c)$ be a braided monoidal category. The following conditions are equivalent:*
(i) For every U, V, $W \in \mathcal{C}$ the following diagram is commutative:

$$
\begin{tikzcd}
 & (U \otimes V) \otimes W \arrow[dl, "c_{U,V} \otimes id_W"'] \arrow[dr, "a_{U,V,W}"] & \\
(V \otimes U) \otimes W \arrow[d, "a_{V,U,W}"'] & & U \otimes (V \otimes W) \arrow[d, "id_U \otimes c_{V,W}"] \\
V \otimes (U \otimes W) \arrow[d, "id_V \otimes c^{-1}_{W,U}"'] & & U \otimes (W \otimes V) \arrow[d, "a^{-1}_{U,W,V}"] \\
V \otimes (W \otimes U) \arrow[d, "a^{-1}_{V,W,U}"'] & & (U \otimes W) \otimes V \arrow[d, "c^{-1}_{W,U} \otimes id_V"] \\
(V \otimes W) \otimes U \arrow[d, "c_{V,W} \otimes id_U"'] & & (W \otimes U) \otimes V \arrow[d, "a_{W,U,V}"] \\
(W \otimes V) \otimes U \arrow[dr, "a_{W,V,U}"'] & & W \otimes (U \otimes V) \arrow[dl, "id_W \otimes c_{U,V}"] \\
 & W \otimes (V \otimes U) &
\end{tikzcd}
$$

(ii) For every U, V, $W \in \mathcal{C}$ the following diagram is commutative:

$$
\begin{tikzcd}
 & (U \otimes V) \otimes W \arrow[dl, "c_{V,U} c_{U,V} \otimes id_W"'] \arrow[dr, "a_{U,V,W}"] & \\
(U \otimes V) \otimes W \arrow[d, "a_{U,V,W}"'] & & U \otimes (V \otimes W) \arrow[d, "id_U \otimes c_{W,V} c_{V,W}"] \\
U \otimes (V \otimes W) \arrow[d, "id_U \otimes c_{W,V} c_{V,W}"'] & & U \otimes (V \otimes W) \arrow[d, "a^{-1}_{U,V,W}"] \\
U \otimes (V \otimes W) \arrow[dr, "a^{-1}_{U,V,W}"'] & & (U \otimes V) \otimes W \arrow[dl, "c_{V,U} c_{U,V} \otimes id_W"] \\
 & (U \otimes V) \otimes W &
\end{tikzcd}
$$

Proof. Take U, V, $W \in \mathcal{C}$. Using only the fact that $\mathcal{C}$ is a braided category we have

$$
\begin{aligned}
&((c_{V,U} c_{U,V}) \otimes id_W) a^{-1}_{U,V,W} (id_U \otimes (c_{W,V} c_{V,W})) a_{U,V,W} \\
&\quad = (c_{V,U} \otimes id_W) a^{-1}_{V,U,W} (id_V \otimes c^{-1}_{U,W}) [(id_V \otimes c_{U,W}) a_{V,U,W} (c_{U,V} \otimes id_W)] \\
&\qquad a^{-1}_{U,V,W} (id_U \otimes c_{W,V} c_{V,W}) a_{U,V,W} \\
&\quad = (c_{V,U} \otimes id_W) a^{-1}_{V,U,W} (id_V \otimes c^{-1}_{U,W}) [a_{V,W,U} c_{U,V \otimes W} a_{U,V,W}] a^{-1}_{U,V,W}
\end{aligned}
$$

$$\begin{aligned}
& (id_U \otimes c_{W,V} \circ c_{V,W}) a_{U,V,W} \\
= \quad & (c_{V,U} \otimes id_W) a^{-1}_{V,U,W} (id_V \otimes c^{-1}_{U,W}) a_{V,W,U} (c_{W,V} \otimes id_U)(c_{V,W} \otimes id_U) c_{U,V\otimes W} a_{U,V,W},
\end{aligned}$$

$$\begin{aligned}
& a^{-1}_{U,V,W}(id_U \otimes c_{W,V})(id_U \otimes c_{V,W}) a_{U,V,W} (c_{V,U} \otimes id_W)(c_{U,V} \otimes id_W) \\
= \quad & a^{-1}_{U,V,W}(id_U \otimes c_{W,V}) a_{U,W,V} c_{V,U\otimes W} a_{V,U,W} (c_{U,V} \otimes id_W) \\
= \quad & a^{-1}_{U,V,W}(id_U \otimes c_{W,V}) a_{U,W,V} c_{V,U\otimes W} (id_V \otimes c^{-1}_{U,W})(id_V \otimes c_{U,W}) a_{V,U,W} (c_{U,V} \otimes id_W) \\
= \quad & a^{-1}_{U,V,W}(id_U \otimes c_{W,V}) a_{U,W,V} (c^{-1}_{U,W} \otimes id_V) c_{V,W\otimes U} (id_V \otimes c_{U,W}) a_{V,U,W} (c_{U,V} \otimes id_W) \\
= \quad & a^{-1}_{U,V,W}(id_U \otimes c_{W,V}) a_{U,W,V} (c^{-1}_{U,W} \otimes id_V) a^{-1}_{W,U,V} (id_W \otimes c_{V,U}) a_{W,V,U} (c_{V,W} \otimes id_U) a^{-1}_{V,W,U} \\
& (id_V \otimes c_{U,W}) a_{V,U,W} (c_{U,V} \otimes id_W) \\
= \quad & a^{-1}_{U,V,W}(id_U \otimes c_{W,V}) a_{U,W,V} (c^{-1}_{U,W} \otimes id_V) a^{-1}_{W,U,V} (id_W \otimes c_{V,U}) a_{W,V,U} (c_{V,W} \otimes id_U) a^{-1}_{V,W,U} \\
& a_{V,W,U} c_{U,V\otimes W} a_{U,V,W} \\
= \quad & a^{-1}_{U,V,W}(id_U \otimes c_{W,V}) a_{U,W,V} (c^{-1}_{U,W} \otimes id_V) a^{-1}_{W,U,V} (id_W \otimes c_{V,U}) a_{W,V,U} \\
& (c_{V,W} \otimes id_U) c_{U,V\otimes W} a_{U,V,W}.
\end{aligned}$$

This means that the condition (ii) holds if and only if

$$\begin{aligned}
& (c_{V,U} \otimes id_W) a^{-1}_{V,U,W} (id_V \otimes c^{-1}_{U,W}) a_{V,W,U} (c_{W,V} \otimes id_U) \\
& \qquad = a^{-1}_{U,V,W}(id_U \otimes c_{W,V}) a_{U,W,V} (c^{-1}_{U,W} \otimes id_V) a^{-1}_{W,U,V} (id_W \otimes c_{V,U}) a_{W,V,U},
\end{aligned}$$

and this condition is obviously equivalent with (i). □

Definition 5.9 *We say that a braided monoidal category $(\mathcal{C}, \otimes, I, a, l, r, c)$ is pseudosymmetric if it satisfies any of the two equivalent conditions from Proposition 5.8.*

Remark 5.10 *If $(\mathcal{C}, \otimes, I, a, l, r, c)$ is a pseudosymmetric braided monoidal category and V is an object in $\mathcal{C}$, then $c_{V,V}$ is a pseudosymmetric Yang-Baxter operator on V.*

Lemma 5.11 *If the braided category $\mathcal{C}$ is pseudosymmetric then $Br(\mathcal{B}, \mathcal{C}) \cong Br(\mathcal{PS}, \mathcal{C})$.*

Proof. The isomorphism is induced by $\pi : \mathcal{B} \to \mathcal{PS}$. More precisely, we have

$$\pi^* : Br(\mathcal{PS}, \mathcal{C}) \to Br(\mathcal{B}, \mathcal{C}), \quad \pi^*(G) = G \circ \pi.$$

Because $\pi_n : B_n \to PS_n$ is surjective and the category $\mathcal{C}$ is pseudosymmetric, any functor $F \in Br(\mathcal{B}, \mathcal{C})$ is of the form $F = G \circ \pi$ for some unique $G \in Br(\mathcal{PS}, \mathcal{C})$. □

As a consequence of this and Theorem 5.7 we obtain the second universality property of $\mathcal{PS}$:

Theorem 5.12 *For a pseudosymmetric braided category $\mathcal{C}$, the functor $\widetilde{\Theta'} : Br(\mathcal{PS}, \mathcal{C}) \to \mathcal{C}$ defined by $\widetilde{\Theta'}(G) = G(1)$ is an equivalence of categories.*

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